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# Solution of the One-Dimensional Linear Boltzmann Equation for Charged Maxwellian Particles in an External Field

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The one-dimensional linear homogeneous Boltzmann equation is solved for a binary mixture of quasi-Maxwellian particles in the presence of a time-dependent external field. It is assumed that the charged particles move in a bath of neutral scatterers. The neutral scatterers are in thermal equilibrium and the concentration of the charged particles is low enough to neglect collisions between them. Two cases are considered in detail, the constant and the periodic external field. The quantities calculated are the equilibrium and the stationary distribution function, respectively, from which any desired property can be derived. The solution of the Boltzmann equation for Maxwellian particles can be reduced to the solution of the so-called cold gas equation by employing the one-dimensional variant of a convolution theorem due to Wannier. The two limiting cases, the Lorentz gas ($m_A \to 0$) and the Rayleigh gas ($m_A \to \infty$) are treated explicitly. Furthermore, by computing the central moments, the deviations from the Gaussian approximation are discussed, and in particular the large-velocity tails are evaluated.

**KEY WORDS:** One-dimensional Boltzmann equation; time-dependent external field; Maxwell gas.

## 1. INTRODUCTION

We consider a one-dimensional binary mixture of quasi-Maxwellian particles with arbitrary mass ratio where the charged component is moving in the bath of the neutral component and interacting with a time-dependent external field. The system is assumed to be spatially homogeneous, and the quantity of interest is the time-dependent velocity distribution of the charged particles and quantities derived therefrom.

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The reason for considering this simple model is that it allows the calculation of both nonequilibrium and equilibrium properties in an analytic form, starting from the linear Boltzmann equation. For this equation to be a valid description the following assumptions are made: (1) Only binary collisions between charged and neutral particles take place. (2) The neutral particles are assumed to be in thermal equilibrium and hold their velocity distribution constant in time. (3) The concentration of the charged particles is low enough to neglect collisions between them.

The assumption of a Maxwellian interaction, though an artifice in one dimension, can be justified for three reasons: First, it makes the collision probability independent of the velocity of the particle, thereby making the equation amenable to analytic treatment. Second, the results can be compared with calculations for the more realistic hard-sphere interaction. Third, in three dimensions the Maxwell model can be justified by the fact that at large enough distances the polarization potential for the ion-neutral interaction varies proportional to $r^{-4}$, leading to a velocity-independent collision frequency.

Two special cases of this model are also considered in this paper. The first is the Lorentz model, where light charged particles move in a bath of heavy neutral scatterers. The second is the Rayleigh model, i.e., the motion of heavy charged particles in a bath of light neutrals.

In the case of equal masses the system turns out to be of special simplicity and a solution can be found for an arbitrary time-dependent external field. This system is related to an idealized moment charge transfer model, where an ion transfers its charge to the neutral particle upon collision, but no momentum or energy is transferred.

The mathematically more involved system of hard rods in one dimension with a constant external field has been studied by Piasecki and Wajnryb,[1] Piasecki,[2] and Gervois and Piasecki.[3] In the last paper the equilibrium velocity distribution of hard rods is obtained for a system of equal masses.

In the following sections we only quote the work of authors directly related to our topic. The general development of the subject can best be traced in the book of Chapman and Cowling[4] and the publications by Kihara,[5] Wannier,[6] Kumar and Robson,[7] Whealton and Mason,[8] McDaniel and Mason,[9] and Kumar *et al.*[10] and references quoted therein.

The paper is organized as follows: In Section 2 a convolution theorem is shown for the linear Boltzmann equation with Maxwellian interaction. This theorem allows us to represent the distribution function in a series of the moments of the so-called cold gas equation. In Sections 3 and 4 the Boltzmann equation is solved for equal masses and arbitrary mass ratios, respectively, and the case of a constant external field is investigated in more detail. In Section 5 the results for the two limiting cases, the Lorentz-limit ($m_A \to 0$) and the Rayleigh limit ($m_A \to \infty$), are considered. In Section 6 we study the case of a periodic external field both for finite mass ratios and for the Lorentz and Rayleigh limits; concluding remarks are made in Section 7.

Note that throughout this paper we use the term *equilibrium* distribution if the limit $\tau \to \infty$ exists, whereas the term *stationary* distribution is used if the distribution function becomes periodic in the long-time limit (derived by the limit $\lim_{\tau_0 \to -\infty}$, where $\tau_0$ is the starting time).

## 2. BOLTZMANN EQUATION FOR CHARGED PARTICLES

We consider the motion of charged particles of mass $m_A$ and charge $q$ in a bath of neutral particles with mass $m_B$ under the influence of a time-dependent external electric field $E(t)$. Assuming that collisions between charged particles can be neglected ($n_A \approx 0$) and restricting ourselves to the spatially homogeneous case, we can calculate the velocity distribution function of the charged particles via the linear Boltzmann equation

$$\begin{aligned}\frac{\partial}{\partial t} h(v, t) + \frac{q}{m_A} E(t) \frac{\partial}{\partial v} h(v, t)\\ = n_B \int [h(v', t) f_B(v_1') - h(v, t) f_B(v_1)] \, |v_1 - v| \, \sigma \, dv_1 \qquad (2.1)\end{aligned}$$

where the postcollisional velocities $v'$ and $v_1'$ for a one-dimensional interaction are given by

$$v' = \frac{m_A - m_B}{m_A + m_B} v + \frac{2m_B}{m_A + m_B} v_1 \qquad (2.2a)$$

$$v_1' = \frac{2m_A}{m_A + m_B} v - \frac{m_A - m_B}{m_A + m_B} v_1 \qquad (2.2b)$$

Furthermore, $n_B$ is the number density of the bath and $f_B$ denotes the Maxwell–Boltzmann equilibrium distribution function

$$f_B(v) = (1/\pi^{1/2} v_{T_B}) \exp(-v^2/v_{T_B}^2), \qquad v_{T_B}^2 = 2kT/m_B \qquad (2.3)$$

In analogy to the three-dimensional case we *assume* the collision cross section $\sigma$ to be inversely proportional to the relative speed, i.e.,

$$|v_1 - v| \, \sigma = \sigma_0 \qquad (2.4)$$

where $\sigma_0$ is a velocity-independent interaction constant. Since in three dimensions (2.4) holds for the Maxwellian ($r^{-4}$) interaction law, we use the same terminology in one dimension. As in three dimensions, one calculates the collision cross section in one dimension using the *Stosszahlansatz* and the conservation equations (2.2a), (2.2b). The collision cross section (2.4) is *designed* to produce the same velocity dependence in one and three dimensions for quantities such as the collision frequency and the momentum and energy transfer. As far as the relationship to realistic dynamics is concerned, a comparison between the one-[11,12] and three-dimensional[13] dynamical self-structure factors for the Maxwellian and the hard core interactions shows close similarity. Furthermore a one-dimensional Maxwellian gas mixture exhibits a realistic dynamics when approaching the thermal equilibrium.[14,15]

With this simplification the Boltzmann equation reads

$$\frac{\partial}{\partial \tau} h + a(\tau) \frac{\partial}{\partial v} h + h = \int f_B(v_1')\, h(v', \tau)\, dv_1 \tag{2.5}$$

where we have introduced the scaled time and acceleration

$$\tau = n_B \sigma_0 t \tag{2.6a}$$

$$a(\tau) = \frac{q}{n_B \sigma_0 m_A v_{T_A}} E(t) \tag{2.6b}$$

and the velocities are scaled with $v_{T_A}$. Note that $n_B \sigma_0$ is the (velocity-independent) collision frequency.

The search for a solution of Eq. (2.5) is considerably simplified by a theorem shown by Wannier[6] for the three-dimensional case: Let $h_c(v, \tau)$ be the solution of the "cold gas equation"

$$\frac{\partial}{\partial \tau} h_c + a(\tau) \frac{\partial}{\partial v} h_c + h_c = \int \delta(v_1')\, h_c(v', \tau)\, dv_1 \tag{2.7}$$

Then the solution of Eq. (2.5) is given by the convolution of the cold gas solution and the Maxwell–Boltzmann equilibrium distribution of species $A$

$$h(v, \tau) = \int h_c(u, \tau)\, f_A(v-u)\, du \tag{2.8}$$

provided (2.8) holds for the initial condition, too. Equation (2.8) is called the cold gas equation, because it is obtained as the limit of the Boltzmann equation (2.5) for $T \to 0$. It should be noted that this theorem holds for a constant collision frequency only (Maxwellian particles).

In order to prove this theorem, we first evaluate the integral on the right-hand side of Eq. (2.7) with the aid of the conservation equations (2.2a) (2.2b). In the case of equal masses ($m_A = m_B$) we have $v' = v_1$ and $v_1' = v$ and therefore the cold gas equation reduces to

$$\frac{\partial}{\partial \tau} h_c + a(\tau) \frac{\partial}{\partial v} h_c + h_c = \delta(v) \tag{2.9}$$

Clearly, the solution of the Boltzmann equation for equal masses

$$\frac{\partial}{\partial \tau} h + a(\tau) \frac{\partial}{\partial v} h + h = f(v) \tag{2.10}$$

(where $f := f_A \equiv f_B$) is given by the convolution (2.8).

For $m_A \neq m_B$ the cold gas equation reads

$$\frac{\partial}{\partial \tau} h_c + a(\tau) \frac{\partial}{\partial v} h_c + h_c = \frac{1}{|\Delta|} h_c(\Delta^{-1} v, \tau) \tag{2.11}$$

where

$$\Delta := \frac{m_A - m_B}{m_A + m_B} \tag{2.12}$$

Writing $u$ instead of $v$ in Eq. (2.11), multiplying it with $f_A(v-u)$, and integrating over $u$ yields the same left-hand side as in Eq. (2.5). For the right-hand side we have to show that

$$\frac{1}{|\Delta|} \int h_c(\Delta^{-1} u, \tau) f_A(v-u)\, du = \int f_B(v_1') \int h_c(u, \tau) f_A(v'-u)\, du\, dv_1 \tag{2.13}$$

and this can be easily verified by virtue of

$$f_B(v_1') f_A(v'-u) = f_A(v - \Delta u) f_B(v_1 - (1+\Delta) u) \tag{2.14}$$

and therefore the convolution theorem for the one-dimensional Maxwell gas is proven.

This convolution theorem allows us to represent the distribution function $h$ in terms of the moments of the cold gas solution $h_c$,

$$\langle v^n \rangle_c(\tau) := \int v^n h_c(v, \tau)\, dv \tag{2.15}$$

In order to derive this representation, we take the Fourier transform of

Eq. (2.8) and note that $\tilde{h}_c$ can be expressed as a Taylor series with $\langle v^n \rangle_c(\tau)$ as coefficients

$$\tilde{h}(w, \tau) = \tilde{f}_A(w) \sum_{n=0}^{\infty} \frac{(-iw)^n}{n!} \langle v^n \rangle_c(\tau) \tag{2.16}$$

Transforming back Eq. (2.16) yields for the distribution function

$$h(v, \tau) = f_A(v) \sum_{n=0}^{\infty} \frac{1}{n!} H_n(v) \langle v^n \rangle_c(\tau) \tag{2.17}$$

where $H_n$ are the Hermite polynomials. Putting $\tau = 0$ in (2.17) and using the orthogonality of the Hermite polynomials yields

$$\langle v^n \rangle_c(0) = \frac{1}{2^n} \int h_0(v)\, H_n(v)\, dv \tag{2.18}$$

Equations (2.17) and (2.18) show that the distribution function is known as soon as the cold gas moments are available.

In the case of the Lorentz gas, i.e., for $m_A \to 0$, the cold gas equation (2.11) and the Boltzmann equation are identical ($\Delta = -1$),

$$\frac{\partial}{\partial \tau} h + a(\tau) \frac{\partial}{\partial v} h = h(-v, \tau) - h(v, \tau) \tag{2.19}$$

This can also be seen from Eq. (2.8), since for $m_A \to 0$ the equilibrium distribution $f_A$ tends to a $\delta$-function and therefore we get $h(v, \tau) = h_c(v, \tau)$.

In the next section we are going to solve the Boltzmann equation for equal masses, and it will turn out that the time-dependent distribution function can be expressed in terms of elementary functions, namely as a difference of two error functions.

## 3. SOLUTION OF THE BOLTZMANN EQUATION FOR EQUAL MASSES

In order to solve the cold gas equation for the case that the ions and the bath particles have equal masses ($m_A = m_B$), we take the Fourier transform of Eq. (2.9)

$$\frac{\partial}{\partial \tau} \tilde{h}_c + iwa(\tau)\, \tilde{h}_c + \tilde{h}_c = 1 \tag{3.1}$$

where $\tilde{h}_c(w, \tau) = \int h_c(v, \tau) \exp(-ivw)\, dv$. Solving this first-order ordinary differential equation and transforming back, we get as solution of the cold gas equation

$$h_c(v, \tau) = \int_{\tau_0}^{\tau} e^{-(\tau - \tau')}\, \delta(v - v_a(\tau' \mid \tau))\, d\tau' + e^{-(\tau - \tau_0)}\, h_{c,0}(v - v_a(\tau_0 \mid \tau)) \tag{3.2}$$

where $v_a(\tau_0 \mid \tau)$ is the velocity gained by acceleration due to the external field during the time interval $[\tau_0, \tau]$,

$$v_a(\tau_0 \mid \tau) = \int_{\tau_0}^{\tau} a(\tau')\, d\tau' \tag{3.3}$$

and $h_{c,0}$ denotes the cold gas initial condition. From this result we get the solution of the linear Boltzmann equation for particles with equal masses in a time-dependent external field by applying the convolution theorem derived in the previous section. The result is

$$h(v, \tau) = \frac{1}{\sqrt{\pi}} \int_{\tau_0}^{\tau} e^{-(\tau - \tau')}\, e^{-[v - v_a(\tau' \mid \tau)]^2}\, d\tau' + e^{-(\tau - \tau_0)}\, h_0(v - v_a(\tau_0 \mid \tau)) \tag{3.4}$$

For the special case of a constant external field $a(\tau) \equiv a$ the integration in Eq. (3.4) can be carried out, yielding

$$\begin{aligned} h(v, \tau) = {} & \frac{1}{\pi^{1/2} a}\, e^{-v^2} e^{(1/2a - v)^2} \left[ \operatorname{erf}\left( a\tau + \frac{1}{2a} - v \right) - \operatorname{erf}\left( \frac{1}{2a} - v \right) \right] \\ & + e^{-\tau} h_0(v - a\tau) \end{aligned} \tag{3.5}$$

where we have set $\tau_0 = 0$, and

$$\operatorname{erf}(x) := \int_0^x e^{-y^2}\, dy \tag{3.6}$$

The equilibrium distribution is obtained by letting $\tau \to \infty$ in Eq. (3.5):

$$h^{\mathrm{eq}}(v) = \frac{1}{\pi^{1/2} a}\, e^{-v^2} e^{(1/2a - v)^2} \left[ \frac{\sqrt{\pi}}{2} \operatorname{sgn}(a) - \operatorname{erf}\left( \frac{1}{2a} - v \right) \right] \tag{3.7}$$

This shows that the charged particle distribution does not approach a Maxwell–Boltzmann equilibrium distribution (see Fig. 1).

One of the quantities of primary interest in the study of the relaxation of charged particles is their drift velocity, i.e., the mean velocity in equilibrium. For our special model we get

$$\langle v \rangle := \int v h^{\mathrm{eq}}(v)\, dv = a \tag{3.8a}$$

or, in unscaled variables [see (2.6a), (2.6b)],

$$\langle v \rangle = (q/nm\sigma_0)\, E \tag{3.8b}$$

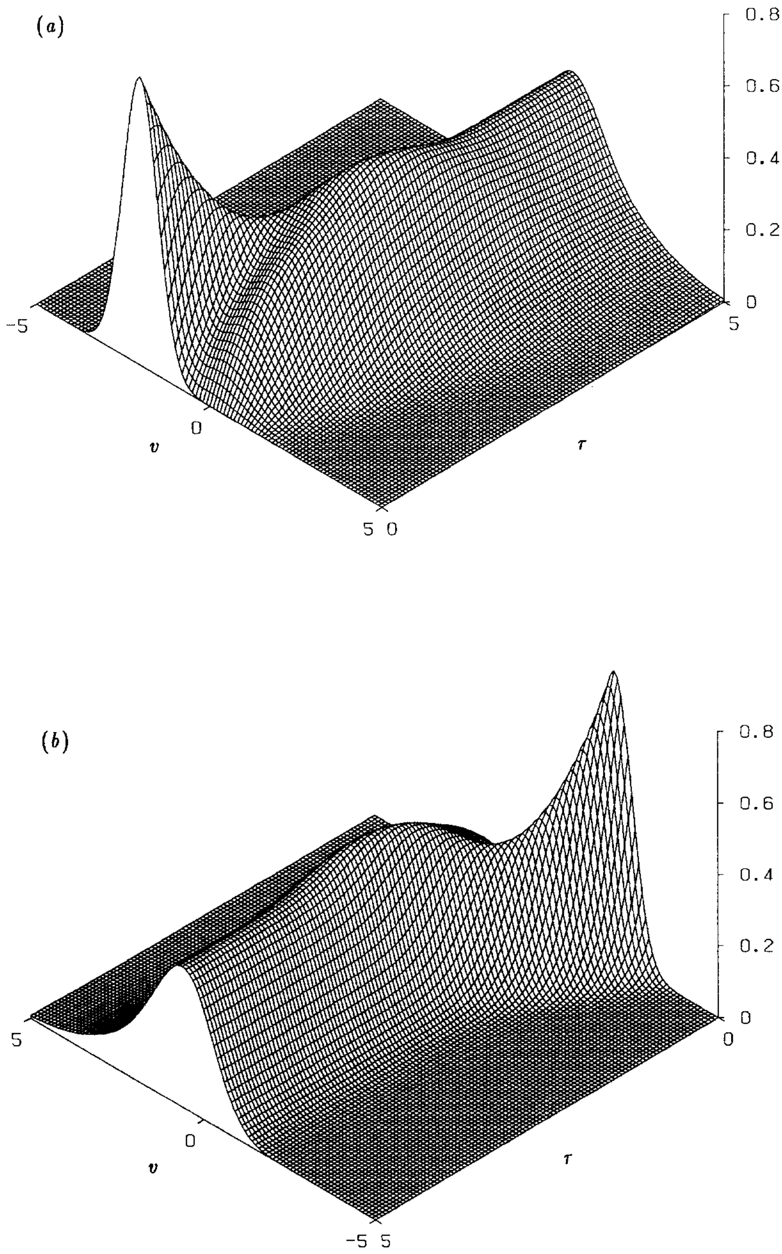


Fig. 1. (a) Time-dependent velocity distribution function for a field strength $a = 1$, $m_A = m_B$, and a shifted Gaussian initial distribution ($\langle v \rangle = -2$, $\sigma = 0.5$); (b) same as (a), seen from the long-time side. The approach to a non-Gaussian equilibrium distribution can be clearly seen.

showing that the mobility $K$ is independent of the external field

$$K = q/nm\sigma_0 \tag{3.9}$$

The independence of the mobility from the external field is a special feature of the Maxwellian interaction law. For the "more realistic" hard rod gas the drift velocity, and consequently the mobility, shows a complicated dependence on the external field (see ref. 3). In the weak and strong field limits this dependence is given by

$$K = \begin{cases} \dfrac{q}{nmv_T\gamma} & \text{for} \quad E \to 0 \\ \left(\dfrac{2q}{\pi nm}\dfrac{1}{E}\right)^{1/2} & \text{for} \quad E \to \infty \end{cases}$$

where

$$\gamma := \frac{\pi}{2}\int_0^\infty \frac{dx}{[e^{-x^2} + 2x\,\mathrm{erf}(x)]^2} = 1.5234789...$$

showing that the mobility of charged hard rods tends to zero for $E \to \infty$.

The shape of the distribution function can be best examined with the aid of the central moments of $h(v, \tau)$. To calculate these moments, it is advantageous to start from the integral representation (3.4). Although it does not pose any problem to compute the moments as a function of time, we restrict ourselves to the central moments of the equilibrium distribution,

$$\langle (v - \langle v \rangle)^n \rangle = \int_0^\infty e^{-\tau'} \frac{1}{\sqrt{\pi}} \int (v-a)^n e^{-(v-a\tau')^2} \, dv \, d\tau' \tag{3.10}$$

This integral can be easily evaluated and yields expressions for the moments in terms of a double sum. However, we refrain from writing them down, since in the next section we will derive more general expressions, valid also for different masses.

## 4. SOLUTION OF THE BOLTZMANN EQUATION FOR DIFFERENT MASSES

In order to solve the Boltzmann equation for different masses we only have to find expressions for the cold gas moments $\langle v^n \rangle_c(\tau)$ (see Section 2). Assuming that $v^n h_c \to 0$ as $|v| \to \infty$ and integrating by parts, we get the following set of differential equations for the cold gas moments:

$$\frac{d}{d\tau}\langle v^n \rangle_c + \mu_n \langle v^n \rangle_c = na(\tau)\langle v^{n-1} \rangle_c(\tau), \qquad n \geqslant 1 \tag{4.1}$$

with

$$\mu_n := 1 - \Delta^n \tag{4.2}$$

and the starting condition $\langle v^0 \rangle_c(\tau) \equiv 1$. The solution of Eq. (4.1) is given by

$$\langle v^n \rangle_c(\tau) = n \int_{\tau_0}^{\tau} e^{-\mu_n(\tau - \tau')} a(\tau') \langle v^{n-1} \rangle_c(\tau')\, d\tau' + e^{-\mu_n(\tau - \tau_0)} \langle v^n \rangle_c(0) \tag{4.3}$$

where $\langle v^n \rangle_c(0)$ is connected with the initial distribution $h_0$ via Eq. (2.18).

For the special case of a constant external field $a(\tau) \equiv a$ these moments can be calculated explicitly. Taking into account that the integral in Eq. (4.3) is a convolution integral, the moments are most easily evaluated by taking the Laplace transform of Eq. (4.3). The final result is ($\tau_0 = 0$)

$$\langle v^n \rangle_c(\tau) = \sum_{k=0}^{n} \frac{n!}{k!} \langle v^k \rangle_c(0)\, a^{n-k} \sum_{j=k}^{n} A_{n,k,j} e^{-\mu_j \tau} \tag{4.4}$$

with

$$A_{n,k,j} = \prod_{\substack{l=k \\ l \neq j}}^{n} \frac{1}{\Delta^j - \Delta^l} \tag{4.5}$$

By using Eq. (2.17), we finally get for the time-dependent distribution function

$$h(v, \tau) = f_A(v) \sum_{n=0}^{\infty} H_n(v) \sum_{k=0}^{n} \frac{1}{k!} \langle v^k \rangle_c(0)\, a^{n-k} \sum_{j=k}^{n} A_{n,k,j} e^{-\mu_j \tau} \tag{4.6}$$

For $\tau \to \infty$ we get the equilibrium distribution ($\mu_0 = 0$)

$$h^{\text{eq}}(v) = f_A(v) \sum_{n=0}^{\infty} \frac{a^n}{(\Delta; \Delta)_n} H_n(v) \tag{4.7}$$

where we have introduced the commonly used abbreviation

$$(z; q)_n := \prod_{l=0}^{n-1} (1 - zq^l) \qquad \text{and} \qquad (z; q)_0 := 1 \tag{4.8}$$

Since the representation (4.7) also holds for equal masses ($\Delta = 0$), we get by comparison with Eq. (3.7) the following interesting expansion:

$$\frac{1}{a} e^{(1/2a - v)^2} \left[ \frac{\sqrt{\pi}}{2} \operatorname{sgn}(a) - \operatorname{erf}\left( \frac{1}{2a} - v \right) \right] = \sum_{n=0}^{\infty} a^n H_n(v) \tag{4.9}$$

For the actual computation of the equilibrium distribution for different masses the infinite sum (4.7) poses considerable difficulties, especially for $|a| > 1$. Therefore, we derive an alternative representation, which is well suited for numerical computations, but is also interesting by itself. A theorem by Euler[16] states that for $|z| < 1$ and $|q| < 1$

$$\sum_{k=0}^{\infty} \frac{z^k q^{k(k-1)/2}}{(q; q)_k} = (-z; q)_\infty \tag{4.10}$$

Putting $q = \Delta$ and $z = -\Delta^{n+1}$, Eq. (4.10) becomes

$$\frac{1}{(\Delta; \Delta)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \Delta^{kn} \Delta^{k(k+1)/2}}{(\Delta; \Delta)_k} = \frac{1}{(\Delta; \Delta)_n} \tag{4.11}$$

Replacing $1/(\Delta; \Delta)_n$ in Eq. (4.7) by the left-hand side of Eq. (4.11) and interchanging the summations, one finds

$$h^{\mathrm{eq}}(v) = \frac{1}{(\Delta; \Delta)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \Delta^{k(k+1)/2}}{(\Delta; \Delta)_k} h_1^{\mathrm{eq}}(v; a\Delta^k) \tag{4.12}$$

where $h_1^{\mathrm{eq}}(v; a)$ denotes the equilibrium distribution function for equal masses with constant external field $a$ given by Eq. (3.7). Equation (4.12) is well suited for an efficient numerical evaluation of the equilibrium distribution function for different masses.

From Eq. (4.7) the drift velocity can be easily calculated and is given by

$$\langle v \rangle = \frac{a}{\mu_1} = \frac{a}{1-\Delta} = a\,\frac{m_A + m_B}{2m_B} \tag{4.13}$$

showing, as in the case of equal masses, that the mobility is independent of the external field.

In order to investigate the shape of the equilibrium distribution function, we calculate its central moments. This can be most easily done by expressing the powers of the velocity in terms of Hermite polynomials (see Appendix) and then using their orthogonality. For the even moments we find

$$\left\langle \left(v - \frac{a}{\mu_1}\right)^{2m} \right\rangle = \frac{(2m)!}{2^{2m} m!} \left(1 + m\,\frac{2a^2}{\mu_2}\right) + \frac{(2m)!}{2^{2m}} \sum_{k=2}^{m} \frac{(2a)^{2k}}{(m-k)!} \times \sum_{l=0}^{2k} \frac{(-1)^l}{l!\, \mu_1^l (\Delta; \Delta)_{2k-l}} \tag{4.14}$$

Putting $m=1$, we obtain the variance of the equilibrium distribution function

$$\sigma^2 := \langle v^2 \rangle - \langle v \rangle^2 = \frac{1}{2} + \frac{a^2}{\mu_2} \tag{4.15}$$

For the odd moments we get ($m \geqslant 2$)

$$\left\langle \left( v - \frac{a}{\mu_1} \right)^{2m-1} \right\rangle = \frac{(2m-1)!}{2^{2m-1}} \sum_{k=2}^{m} \frac{(2a)^{2k-1}}{(m-k)!} \sum_{l=0}^{2k-1} \frac{(-1)^l}{l!\, \mu_1^l (\Delta; \Delta)_{2k-l-1}} \tag{4.16}$$

Putting $m=2$, we find that the equilibrium distribution is skewed:

$$\langle (v - a/\mu_1)^3 \rangle = 2a^3/\mu_3 \tag{4.17}$$

However, this skewness, as well as all higher odd moments, vanishes with order $a^3$ or higher as $a \to 0$. Moreover, it turns out that in the low-field limit the equilibrium distribution function can be approximated by a Gaussian distribution with mean $a/\mu_1$ [cf. Eq. (4.13)] and variance $\frac{1}{2} + a^2/\mu_2$ [cf. Eq. (4.15)],

$$h^G(v) = \frac{1}{[\pi(1 + 2a^2/\mu_2)]^{1/2}} \exp\left[ -\frac{(v - a/\mu_1)^2}{1 + 2a^2/\mu_2} \right] \tag{4.18}$$

While the odd moments of $h^G$ vanish identically (which means that they are identical to the odd moments of $h^{eq}$ up to order $a^2$), the even moments of $h^G$ are given by

$$\begin{aligned} \left\langle \left( v - \frac{a}{\mu_1} \right)^{2m} \right\rangle &= \frac{(2m)!}{2^{2m} m!} \left( 1 + \frac{2a^2}{\mu_2} \right)^m \\ &= \frac{(2m)!}{2^{2m} m!} \left( 1 + m \frac{2a^2}{\mu_2} \right) + O(a^4), \qquad a \to 0 \end{aligned} \tag{4.19}$$

which shows that in the low-field limit ($a \to 0$) the even moments coincide up to order $a^3$ with the moments of the exact equilibrium distribution [compare Eq. (4.14)]. For the quality of the Gaussian approximation in the low-field limit see Fig. 2.

For *large* values of the external field we get from Eqs. (4.14) and (4.16)

$$\left\langle \left( v - \frac{a}{\mu_1} \right)^n \right\rangle = a^n n! \sum_{l=0}^{n} \frac{(-1)^l}{l!\, \mu_1^l (\Delta; \Delta)_{n-l}} + O(a^{n-2}), \qquad a \to \infty \tag{4.20}$$

For *equal masses* ($\Delta = 0$, $\mu_1 = 1$) these are exactly the central moments of the exponential distribution

$$h^E(v) = \Theta(v) \frac{1}{a} e^{-v/a} \tag{4.21}$$

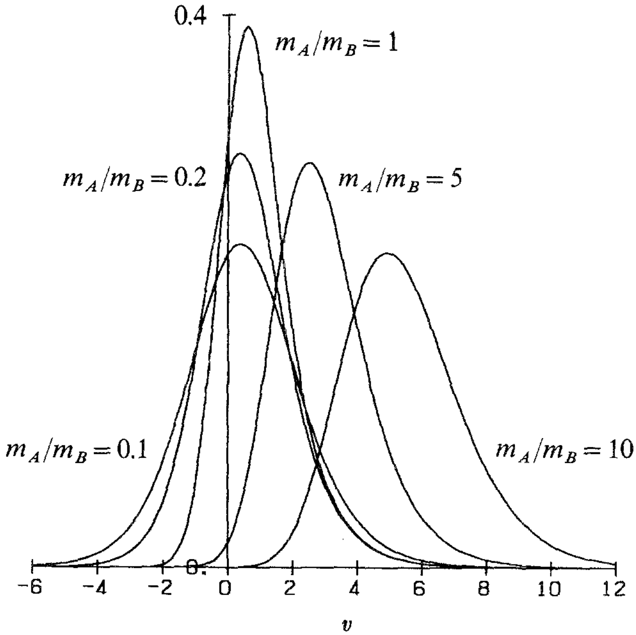


Fig. 2. Velocity equilibrium distributions for a field strength $a = 1$ and different mass ratios $m_A/m_B = 0.1$, 0.2, 1, 5, 10. Note that for reciprocal mass ratios the distribution function look similar in shape, but is shifted in $v$.

where $\Theta$ denotes the unit step function. In Fig. 3 we display the exact distribution function [cf. Eq. (3.7)] and the approximation (4.21) as a function of $v/a$ for different values of $a$. It shows that for large $v$ the presence of an external field changes the large-velocity tail qualitatively from $\exp(-v^2)$ (no field) to $\exp(-v/a)$ (large field).

For the case of *different masses* the approximate distribution can be found by replacing $h_1(v; a\Delta^k)$ by $(a\Delta^k)^{-1} \exp(v/a\Delta^k)$ in Eq. (4.12). However, the resulting expression does not reduce to an elementary function and therefore it will not be pursued further.

## 5. THE LORENTZ LIMIT ($m_A \to 0$) AND THE RAYLEIGH LIMIT ($m_A \to \infty$)

In the Lorentz limit, where the ions are scattered on immobile particles of infinite mass, the cold gas equation and the Boltzmann equation are identical and given by Eq. (2.19). In order to solve this equation, we introduce the even and odd parts of the distribution function $h_{ev}(v, \tau)$ and

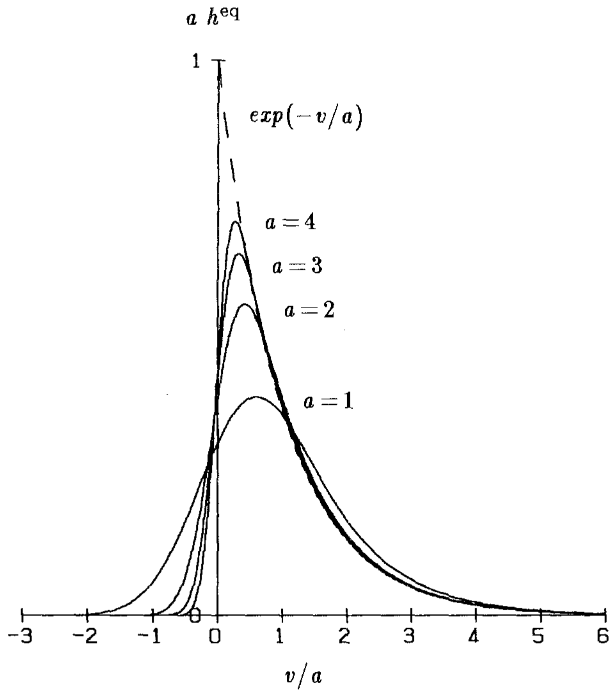


Fig. 3. High-velocity tail $\exp(-v/a)$ (dashed line) of the velocity equilibrium distribution for different field strengths $a = 1, 2, 3, 4$ plotted as function of $v/a$. For $a = 1$ the asymptotic exponential behavior is reached much later than for higher values of $a$.

$h_{od}(v, \tau)$, respectively. Putting $v \to -v$ in Eq. (2.19) yields the following pair of coupled partial differential equations for $h_{ev}$ and $h_{od}$:

$$\frac{\partial}{\partial \tau} h_{ev} + a(\tau) \frac{\partial}{\partial v} h_{od} = 0 \tag{5.1a}$$

$$\frac{\partial}{\partial \tau} h_{od} + a(\tau) \frac{\partial}{\partial v} h_{ev} = -2h_{od}(v, \tau) \tag{5.1b}$$

Taking the Fourier transform in velocity space reduces the system (5.1a), (5.16) to a system of ordinary differential equations

$$\frac{d}{d\tau} \mathbf{h} = -A(\tau)\,\mathbf{h} \tag{5.2a}$$

with

$$\mathbf{h}(w, \tau) = \begin{pmatrix} \tilde{h}_{ev}(w, \tau) \\ \tilde{h}_{od}(w, \tau) \end{pmatrix}, \qquad A(\tau) = \begin{pmatrix} 0 & iwa(\tau) \\ iwa(\tau) & 2 \end{pmatrix} \tag{5.2b}$$

The solution of Eq. (5.2a) is given by

$$\mathbf{h}(w, \tau) = e^{-V(\tau_0 \mid \tau)} \mathbf{h}_0(w) \tag{5.3a}$$

where

$$V(\tau_0 \mid \tau) = \int_{\tau_0}^{\tau} A(\tau')\, d\tau' = \begin{pmatrix} 0 & iwv_a(\tau_0 \mid \tau) \\ iwv_a(\tau_0 \mid \tau) & 2(\tau - \tau_0) \end{pmatrix} \tag{5.3b}$$

and the velocity $v_a(\tau_0 \mid \tau)$ due to the acceleration $a(\tau)$ in $[\tau_0, \tau]$ is given by Eq. (3.3). Furthermore, $\mathbf{h}_0(w) = (\tilde{h}_{\mathrm{ev},0}(w), \tilde{h}_{\mathrm{od},0}(w))^T$, where $\tilde{h}_{\mathrm{ev},0}$ and $\tilde{h}_{\mathrm{od},0}$ denote the Fourier transforms of the even and odd parts of the initial velocity distribution $h_0$, respectively. The matrix exponential in Eq. (5.3a) can be made explicit by diagonalizing the matrix $V(\tau_0 \mid \tau)$; we get

$$\begin{aligned} V(\tau_0 \mid \tau) &= T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1} \\ &= \frac{1}{iwv_a(\lambda_1 - \lambda_2)} \begin{pmatrix} iwv_a & iwv_a \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} -\lambda_2 & iwv_a \\ \lambda_1 & -iwv_a \end{pmatrix} \end{aligned} \tag{5.4a}$$

where the eigenvalues $\lambda_1$ and $\lambda_2$ of $V(\tau_0 \mid \tau)$ are given by

$$\lambda_{1,2} = \tau - \tau_0 \pm [(\tau - \tau_0)^2 - w^2 v_a^2(\tau_0 \mid \tau)]^{1/2} \tag{5.4b}$$

From Eq. (5.4a) we then get

$$e^{-V(\tau_0 \mid \tau)} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{-\lambda_2} - \lambda_2 e^{-\lambda_1} & iwv_a(e^{-\lambda_1} - e^{-\lambda_2}) \\ iwv_a(e^{-\lambda_1} - e^{-\lambda_2}) & \lambda_1 e^{-\lambda_1} - \lambda_2 e^{-\lambda_2} \end{pmatrix} \tag{5.5}$$

The inverse Fourier transformation can be carried out by virtue of

$$\begin{aligned} &\mathscr{F}^{-1} \left( \frac{\exp[t(1 - \alpha^2 w^2)^{1/2}] - \exp[-t(1 - \alpha^2 w^2)^{1/2}]}{(1 - \alpha^2 w^2)^{1/2}} \right) \\ &\quad = \frac{1}{|\alpha|} I_0 \left( \frac{1}{\alpha} (\alpha^2 t^2 - v^2)^{1/2} \right) \Theta(|\alpha|\, t - |v|) \end{aligned} \tag{5.6}$$

where $I_0$ is the modified Bessel function of order zero, and yields for the even and odd parts of the distribution function, respectively,

$$\begin{aligned} h_{\mathrm{ev}}(v, \tau) = h_{\mathrm{ev},0}(v) * \left( \frac{1}{\tau - \tau_0} \frac{\partial}{\partial \beta} + 1 \right) g(\beta; v, \tau) \big|_{\beta = 1} \\ - \bar{a}(\tau_0 \mid \tau)\, h_{\mathrm{od},0}(v) * \frac{\partial}{\partial v} g(1; v, \tau) \end{aligned} \tag{5.7a}$$

and

$$h_{\text{od}}(v,\tau)=h_{\text{od},0}(v)*\left(\frac{1}{\tau-\tau_0}\frac{\partial}{\partial\beta}-1\right)g(\beta;v,\tau)|_{\beta=1}$$

$$-\bar{a}(\tau_0|\tau)\,h_{\text{ev},0}(v)*\frac{\partial}{\partial v}g(1;v,\tau)\tag{5.7b}$$

where the asterisk denotes convolution in velocity space, and $g(\beta; v, \tau)$ is given by

$$g(\beta;v,\tau)=\frac{1}{2\,|\bar{a}(\tau_0|\tau)|}e^{-(\tau-\tau_0)}I_0\left(\frac{1}{\bar{a}(\tau_0|\tau)}[\beta^2v_a^2(\tau_0|\tau)-v^2]^{1/2}\right)$$

$$\times\Theta(\beta\,|v_a(\tau_0|\tau)|-|v|)\tag{5.8}$$

In Eq. (5.8) we have introduced the average acceleration $\bar{a}(\tau_0|\tau)$ in the interval $[\tau_0, \tau]$,

$$\bar{a}(\tau_0|\tau)=\frac{v_a(\tau_0|\tau)}{\tau-\tau_0}=\frac{1}{\tau-\tau_0}\int_{\tau_0}^{\tau}a(\tau')\,d\tau'\tag{5.9}$$

From Eqs. (5.7a), (5.7b) we finally get for the distribution function

$$h(v,\tau)=h_0(v)*\left(\frac{1}{\tau-\tau_0}\frac{\partial}{\partial\beta}-\bar{a}(\tau_0|\tau)\frac{\partial}{\partial v}\right)g(\beta;v,\tau)|_{\beta=1}+h_0(-v)*g(1;v,\tau)\tag{5.10}$$

In order to study the long-time behavior of the distribution function, we calculate the moments $\langle v^n\rangle(\tau)$. To this end, we need the moments of $g(\beta; v, \tau)$,

$$g_k(\beta;\tau):=\int v^k g(\beta;v,\tau)\,dv\tag{5.11}$$

It is easily shown that the odd moments vanish identically, and for the even moments we find, substituting $v=\beta\,|v_a|\cos\theta$,

$$g_{2k}(\beta;\tau)=(\tau-\tau_0)\,v_a^{2k}(\tau_0|\tau)\,\beta^{2k+1}e^{-(\tau-\tau_0)}$$

$$\times\int_0^{\pi/2}I_0(\beta(\tau-\tau_0)\sin\theta)\cos^{2k}\theta\sin\theta\,d\theta\tag{5.12}$$

This integral can be found in integral tables[17] and leads to the following expression for the even moments of $g(\beta; v, \tau)$:

$$g_{2k}(\beta;\tau)=\bar{a}^{2k}(\tau_0|\tau)\frac{(2k)!}{2^{2k+1}k!}[2\beta(\tau-\tau_0)]^{k+1}e^{-(\tau-\tau_0)}$$

$$\times\left[\frac{\pi}{2\beta(\tau-\tau_0)}\right]^{1/2}I_{k+1/2}(\beta(\tau-\tau_0))\tag{5.13}$$

The modified Bessel functions of half-integer order, in turn, can be expressed in terms of elementary functions,[18] which leads to the following representation of $g_{2k}$:

$$g_{2k}(\beta; \tau) = \bar{a}^{2k}(\tau_0 \mid \tau)\, e^{-(\tau - \tau_0)} \gamma_k(\beta(\tau - \tau_0)) \tag{5.14a}$$

where

$$\gamma_k(z) := r_k(z)\, e^{z} - r_k(-z)\, e^{-z} \tag{5.14b}$$

and

$$r_k(z) = \frac{(-1)^k (2k)!}{2^{2k+1}} \sum_{l=0}^{k} \binom{2k-l}{k} \frac{(-2z)^l}{l!} \tag{5.14c}$$

Now it is a simple matter to calculate the moments of the distribution function (5.10). For the even moments we get

$$\begin{aligned} \langle v^{2m} \rangle(\tau) = {} & e^{-(\tau - \tau_0)} \sum_{k=0}^{m} \binom{2m}{2k} \langle v^{2m-2k} \rangle(\tau_0) \\ & \times \bar{a}^{2k}(\tau_0 \mid \tau) \left( \frac{d}{d\tau} + 1 \right) \gamma_k(\tau - \tau_0) \\ & + 2m e^{-(\tau - \tau_0)} \sum_{k=0}^{m-1} \binom{2m-1}{2k} \langle v^{2m-2k-1} \rangle(\tau_0) \\ & \times \bar{a}^{2k+1}(\tau_0 \mid \tau)\, \gamma_k(\tau - \tau_0) \end{aligned} \tag{5.15}$$

and the odd moments are given by

$$\begin{aligned} \langle v^{2m+1} \rangle(\tau) = {} & e^{-(\tau - \tau_0)} \sum_{k=0}^{m} \binom{2m+1}{2k} \langle v^{2m-2k+1} \rangle(\tau_0) \\ & \times \bar{a}^{2k}(\tau_0 \mid \tau) \left( \frac{d}{d\tau} - 1 \right) \gamma_k(\tau - \tau_0) \\ & + (2m+1)\, e^{-(\tau - \tau_0)} \sum_{k=0}^{m} \binom{2m}{2k} \langle v^{2m-2k} \rangle(\tau_0) \\ & \times \bar{a}^{2k+1}(\tau_0 \mid \tau)\, \gamma_k(\tau - \tau_0) \end{aligned} \tag{5.16}$$

In the case of a constant external field, $a(\tau) \equiv a$ $(a > 0)$, the Lorentz gas has been studied by Piasecki,[2] who investigated the runaway effects of electrons for different interaction laws. For a constant external field the distribution function can be written as

$$h(v, \tau) = h_0(v) * \left(\frac{\partial}{\partial \tau} - a\frac{\partial}{\partial v} + 1\right) g(v, \tau) + h_0(-v) * g(v, \tau) \quad (5.17)$$

where $g(v, \tau) \equiv g(1; v, \tau)$ now reads ($\tau_0 = 0$)

$$g(v, \tau) = \frac{1}{2a} e^{-\tau} I_0\left(\frac{1}{a}(a^2\tau^2 - v^2)^{1/2}\right) \Theta(a\tau - |v|) \quad (5.18)$$

Closer inspection of Eq. (5.12) shows that $g(v, \tau)$ is the Green's function of the one-dimensional telegraph equation[19]

$$\frac{\partial^2}{\partial v^2} g = \frac{1}{D}\frac{\partial}{\partial \tau} g + \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} g \quad (5.19)$$

with "diffusion coefficient" $D = a^2/2$ and "propagation speed" $c = a$.

In order to study the long-time limit for the constant field, we have a look at the moments. Putting $m = 0$ in Eq. (5.16), we get for the time-dependent mean velocity ($\tau_0 = 0$)

$$\langle v\rangle(\tau) = \langle v\rangle(0)\, e^{-2\tau} + \tfrac{1}{2}a(1 - e^{-2\tau}) \quad (5.20a)$$

which yields for the drift velocity

$$\langle v\rangle = \lim_{\tau \to \infty} \langle v\rangle(\tau) = a/2 \quad (5.20b)$$

The higher moments, however, do not approach finite limits for $\tau \to \infty$. From Eq. (5.15) we get for the long-time behavior of the even central moments

$$\left\langle\left(v - \frac{1}{2}a\right)^{2m}\right\rangle(\tau) = \frac{(2m)!}{2^{2m}m!}(2a^2\tau)^m + O(\tau^{m-1}), \qquad \tau \to \infty \quad (5.21)$$

The leading term in Eq. (5.21) is the exact expression for the even central moments of the shifted Gaussian distribution,

$$h^{G}(v, \tau) = \frac{1}{(2\pi a^2\tau)^{1/2}} \exp\left[-\frac{(v - a/2)^2}{2a^2\tau}\right] \quad (5.22)$$

describing a simple diffusion process. That this Gaussian distribution is a reasonable approximation to the exact distribution of the Lorentz gas in the long-time limit is further justified by the fact that the odd central moments calculated via Eq. (5.16) grow much slower than the even ones $(m \geqslant 1)$,

$$\left\langle \left(v - \frac{1}{2} a\right)^{2m+1} \right\rangle (\tau) = O(\tau^{m-1}), \qquad \tau \to \infty \tag{5.23}$$

The emergence of a diffusive long-time behavior of the charged particles in the presence of a constant external field is quite surprising, since in the zero-field case the distribution function of the Lorentz gas is given by

$$h(v, \tau) = \tfrac{1}{2}(1 + e^{-\tau})\, h_0(v) + \tfrac{1}{2}(1 - e^{-\tau})\, h_0(-v) \tag{5.24}$$

with the well-defined equilibrium distribution

$$h^{\mathrm{eq}}(v) = \lim_{\tau \to \infty} h(v, \tau) = \tfrac{1}{2}[h_0(v) + h_0(-v)] = h_{\mathrm{ev},0}(v) \tag{5.25}$$

In Fig. 4 we display the decay of a displaced Gaussian initial distribution, seen from both the short- and long-time sides. In Fig. 4b the diffusive Gaussian distribution [cf. Eq. (5.22)] can be clearly recognized.

Finally we want to study the Rayleigh limit, i.e., the limit $m_A \to \infty$ (in one dimension also referred to as the Rayleigh piston problem). By letting $m_A \to \infty$ in the Boltzmann equation (2.5), the collision integral vanishes and only free streaming remains. However, a nontrivial distribution function can be derived from the Boltzmann equation in the Rayleigh limit by rescaling the variables. We put [cf. Eqs. (2.6a), (2.6b)]

$$\tau = \mu_1 n_B \sigma_0 t \tag{5.26a}$$

$$a(\tau) = \frac{q}{\mu_1 n_B \sigma_0 m_A v_{T_A}} E(t) \tag{5.26b}$$

$$v^* = v/v_{T_A}, \qquad v_1^* = v/v_{T_B} \tag{5.26c}$$

where $\mu_1 = 1 - \Delta = 2m_B/(m_A + m_B)$. This scaling corresponds to the usual scaling to the diffusion coefficient $D_{AB} = v_{T_A}^2/2\mu_1$. Inserting Eqs. (5.26a)–(5.26c) into the Boltzmann equation and making use of detailed balance, one can carry out the limit $m_A \to \infty$, yielding the following Fokker–Planck equation (the asterisks have been dropped for convenience):

$$\frac{\partial}{\partial \tau} h + a(\tau) \frac{\partial}{\partial v} h = \frac{\partial}{\partial v}(vh) + \frac{1}{2} \frac{\partial^2}{\partial v^2} h \tag{5.27}$$

For the details of this derivation see, e.g., ref. 13. It should by mentioned that Eq. (5.27) is obtained for every quasi-$r^{-\nu}$ potential. Equation (5.27)

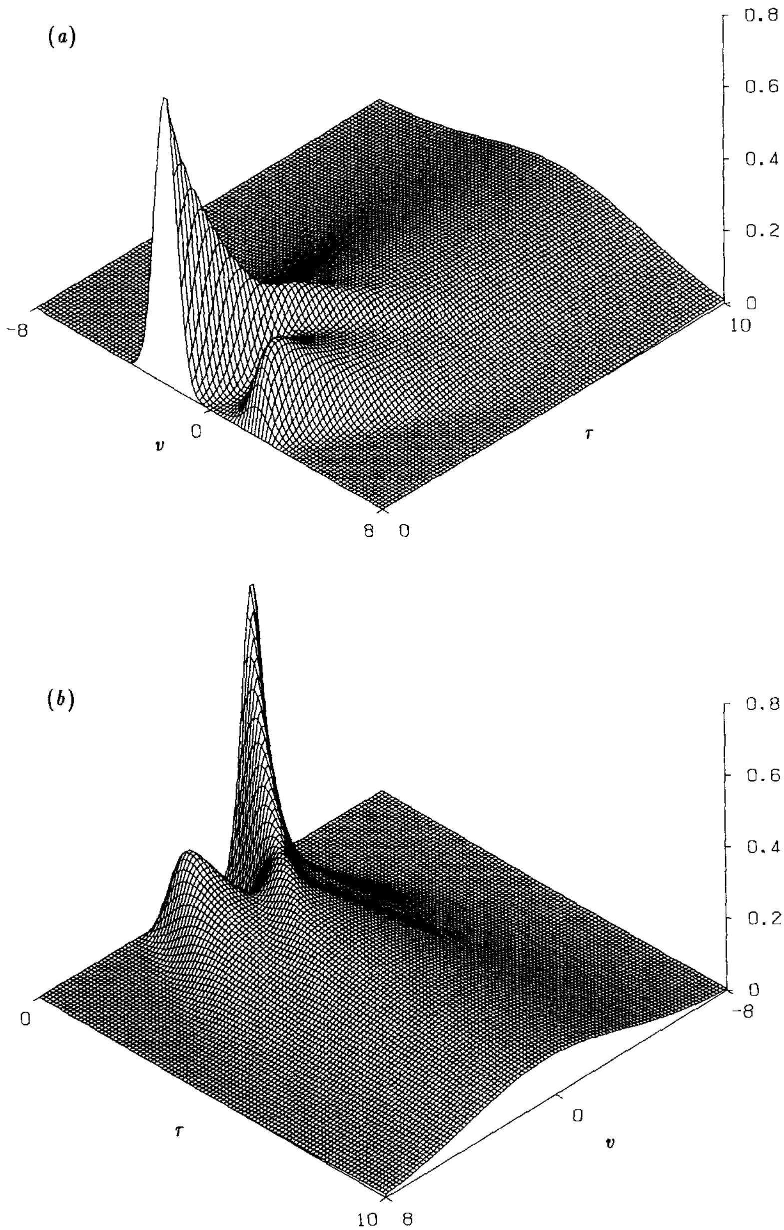


Fig. 4. (a) Approach to equilibrium of a Lorentz gas ($m_A \to 0$) for a field strength $a = 1$ seen from the short-time side. (b) Same as (a), seen from the long-time side. The transition from a distribution determined by collisions to a distribution determined by diffusive behavior can be seen.

has been already derived by Chandrasekhar in his study of Brownian motion in an external field.[20] He also outlined a general solution method for Eq. (5.27). Since the details can be found there, we only write down the result. The time-dependent velocity distribution for a Rayleigh particle in an arbitrary external field with the initial condition $h(v, \tau_0) = \delta(v - v_0)$ is given by

$$h(v, \tau) = \frac{1}{(\pi\{1 - \exp[-2(\tau - \tau_0)]\})^{1/2}} \times \exp\left[-\frac{\{v - v_e(\tau_0 \mid \tau) - v_0 \exp[-(\tau - \tau_0)]\}^2}{1 - \exp[-2(\tau - \tau_0)]}\right] \tag{5.28}$$

where

$$v_e(\tau_0 \mid \tau) = \int_{\tau_0}^{\tau} e^{-(\tau - \tau')} a(\tau')\, d\tau' \tag{5.29}$$

In the special case of a constant external field $a(\tau) \equiv a$ we immediately obtain for the equilibrium distribution ($\tau \to \infty$)

$$h^{\text{eq}}(v) = \frac{1}{\sqrt{\pi}} e^{-(v-a)^2} = f_A(v - a) \tag{5.30}$$

showing that in the case of a Rayleigh particle the shifted Gaussian distribution is exact (see Section 4).

## 6. THE CASE OF A PERIODIC EXTERNAL FIELD

So far we have been mainly concerned with a constant external field. In this section we extend our analysis to the case of a periodic external field,

$$a(\tau) = a \cos \omega\tau \tag{6.1}$$

where $\omega$ is the (scaled) frequency of the field.

In the case of *equal masses* the general expression for the distribution function has been derived in Section 3. For the velocity gained by the acceleration due to the field (6.1) we get [see Eq. (3.3)]

$$v_a(\tau_0 \mid \tau) = \frac{a}{\omega} (\sin \omega\tau - \sin \omega\tau_0) \tag{6.2}$$

Inserting $v_a(\tau_0 \mid \tau)$ into Eq. (3.4) yields the time-dependent velocity distribution function. This function depends also on the initial time $\tau_0$ and not only on the difference $\tau-\tau_0$ (nonautonomous system). We are primarily interested in the shape of the distribution function after a long time ($\tau \gg \tau_0$) when the particular form of the initial distribution is no longer seen. This stationary distribution $h^s$ can be obtained by performing the limit $\tau_0 \to -\infty$,

$$h^s(v, \tau) = \lim_{\tau_0 \to -\infty} h(v, \tau)$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\tau' - \left\{v - \frac{a}{\omega}[\sin \omega\tau - \sin \omega(\tau-\tau')]\right\}^2\right) d\tau' \tag{6.3}$$

This equation contains the two cases of no field and a constant field as limits: for $\omega \to \infty$ we get $h^s = f_A$ and for $\omega \to 0$ we get $h^s = h^{eq}$ [cf. Eq. (3.7)].

In the case of *different masses* we consider again the cold gas moments. For the stationary cold gas moments ($\tau_0 \to -\infty$) we get from Eq. (4.3)

$$\langle v^n \rangle_c(\tau) = n \int_{-\infty}^{\tau} e^{-\mu_n(\tau-\tau')} a \cos \omega\tau' \, \langle v^{n-1} \rangle_c(\tau') \, d\tau' \tag{6.4}$$

We try to solve this recursive equation by the *ansatz*

$$\langle v^n \rangle_c(\tau) = n! \, a^n \sum_{k=0}^{n} B_{n,k} e^{i(n-2k)\omega\tau} \tag{6.5}$$

Insertion into Eq. (6.4) yields the following recursion relations for the frequency- and mass-dependent coefficients $B_{n,k}$

$$B_{0,0} = 1, \qquad B_{n,0} = \frac{1}{2} \frac{B_{n-1,0}}{\mu_n + in\omega}, \qquad B_{n,n} = \frac{1}{2} \frac{B_{n-1,n-1}}{\mu_n - in\omega} \tag{6.6a}$$

and

$$B_{n,k} = \frac{1}{2} \frac{B_{n-1,k} + B_{n-1,k-1}}{\mu_n + i(n-2k)\,\omega}, \qquad k = 1,..., n-1 \tag{6.6b}$$

From Eq. (6.6a) we get immediately

$$B_{n,0} = \frac{1}{2^n} \prod_{l=1}^{n} \frac{1}{\mu_l + il\omega}, \qquad B_{n,n} = \frac{1}{2^n} \prod_{l=1}^{n} \frac{1}{\mu_l - il\omega} = \bar{B}_{n,0} \tag{6.7}$$

where the bar denotes complex conjugation. We have not succeeded in deriving a simple explicit expression for $B_{n,k}$ ($k \neq 0, n$) for arbitrary mass ratios; one can only deduce that $B_{n,n-k} = \bar{B}_{n,k}$. In the case of equal masses, however, the recursion relation (6.6b) can be made explicit, yielding ($\mu_n \equiv 1$ for $n \geqslant 1$)

$$B_{n,k} = \frac{1}{2^n} \binom{n}{k} B_{k,k} B_{n-k,0} = \frac{1}{2^n} \binom{n}{k} \prod_{l=-k}^{n-k} \frac{1}{1 + il\omega} \tag{6.8}$$

The stationary distribution function for arbitrary masses is obtained from Eq. (6.5) by using Eq. (2.17),

$$h^s(v, \tau) = f_A(v) \sum_{n=0}^{\infty} a^n H_n(v) \sum_{k=0}^{n} B_{n,k} e^{i(n-2k)\omega\tau} \tag{6.9}$$

with $B_{n,k}$ given by Eqs. (6.6b) and (6.7). Equation (6.9) shows that the stationary distribution function is periodic with period $2\pi/\omega$. Furthermore, because of $H_n(-v) = (-1)^n H_n(v)$, the stationary distribution has the following additional symmetry:

$$h^s(v, \tau + \pi/\omega) = h^s(-v, \tau) \tag{6.10}$$

In Figs. 5 and 6 we display the stationary distribution function for equal masses for different values of $a$ and $\omega$.

The distribution function of the *Lorentz gas* ($m_A \to 0$) in the case of a periodic external field (6.1) is given by Eqs. (5.9) and (5.10) after inserting Eq. (6.2). In the long-time limit $\tau \to \infty$ (or, alternatively, $\tau_0 \to -\infty$) we have

$$\lim_{\tau_0 \to -\infty} \bar{a}(\tau_0 \mid \tau) = \lim_{\tau \to \infty} \bar{a}(\tau_0 \mid \tau) = 0 \tag{6.11}$$

and therefore we find for the moments from Eqs. (5.15) and (5.16), respectively,

$$\langle v^{2m} \rangle(\tau) \sim \langle v^{2m} \rangle(\tau_0) + O\left(\frac{1}{\tau - \tau_0}\right), \qquad \tau \to \infty \tag{6.12a}$$

and

$$\langle v^{2m+1} \rangle(\tau) \sim O\left(\frac{1}{\tau - \tau_0}\right), \qquad \tau \to \infty \tag{6.12b}$$

This shows that the distribution function approaches an equilibrium with vanishing odd moments, and the even moments are identical to the initial ones. This means that the equilibrium distribution is independent of the external field and is given by Eq. (5.25). However, the decay of the

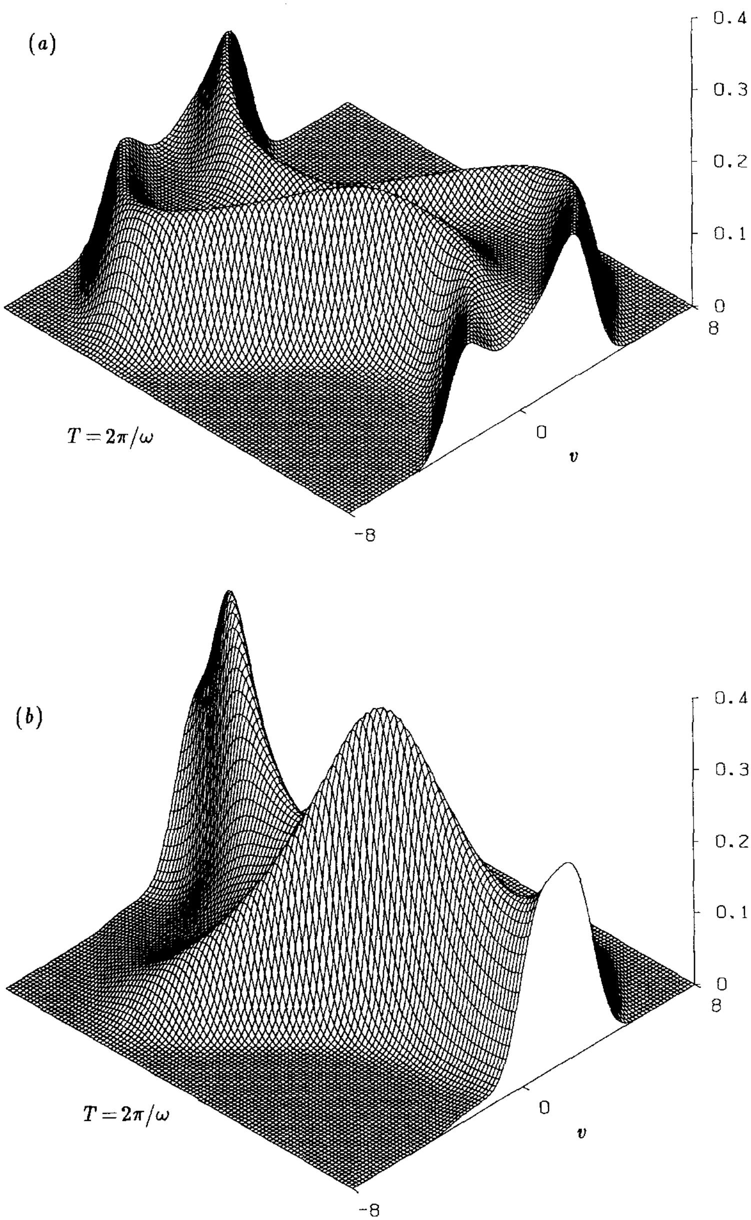


Fig. 5. (a) Stationary velocity distribution function for a periodic external field $a(\tau) = a \cos \omega\tau$ for $a = 24$, $\omega = 8$, and $m_A = m_B$. The high field splits the distribution into a double-peaked structure. (b) Same as (a), for $a = 3$ and $\omega = 1$. The ratio $a/\omega$ is same as in (a), but the field strength is lowered. The split in the distribution is hardly visible.

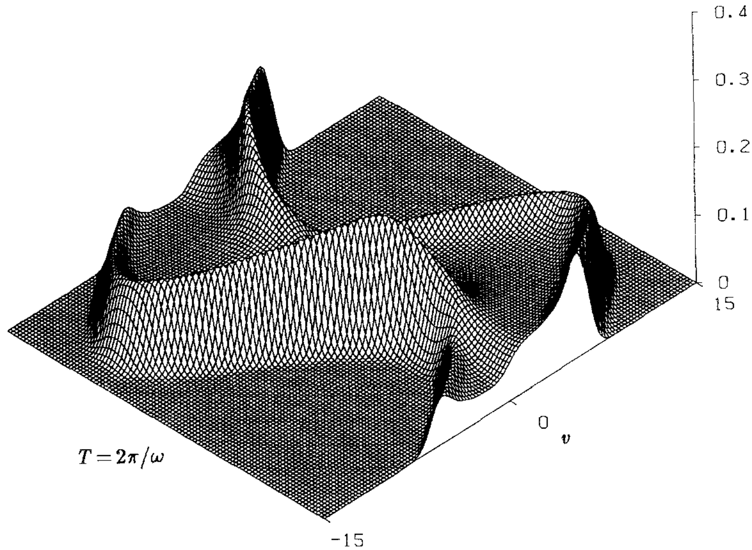


Fig. 6. Stationary velocity distribution function for a periodic external field $a(\tau) = a \cos \omega\tau$ for $a = 24$, $\omega = 4$, and $m_A = m_B$. The field strength is the same as in Fig. 4a, but the frequency is halved. A triple-peak structure becomes visible, and the lower frequency allows wider velocity excursions.

fluctuations of the distribution function due to the periodic external field is of order $(\tau - \tau_0)^{-1}$ and is therefore very slow.

Figures 7a and 7b display the approach to equilibrium for a Lorentz gas for a high and low frequency of the external field.

In the *Rayleigh limit* $(m_A \to \infty)$ we get the distribution function for the periodic external field by inserting the "damped" velocity $v_e(\tau_0 \mid \tau)$, calculated by insertion of Eq. (6.1) into Eq. (5.29), into Eq. (5.28). The stationary distribution is obtained for $\tau_0 \to -\infty$. In this limit $v_e$ reads

$$\lim_{\tau_0 \to -\infty} v_e(\tau_0 \mid \tau) = \frac{a}{1 + \omega^2} (\cos \omega\tau + \omega \sin \omega\tau) \tag{6.13}$$

and therefore the stationary distribution function is given by

$$h^s(v, \tau) = \frac{1}{\sqrt{\pi}} \exp \left\{ - \left[ v - \frac{a}{1 + \omega^2} (\cos \omega\tau + \omega \sin \omega\tau) \right]^2 \right\} \tag{6.14}$$

showing that the stationary distribution function is periodic with period $2\pi/\omega$, but remains Gaussian with variance 1/2, independent of $a$ and $\omega$, and a periodic mean given by Eq. (6.13). An example is displayed in Fig. 8.

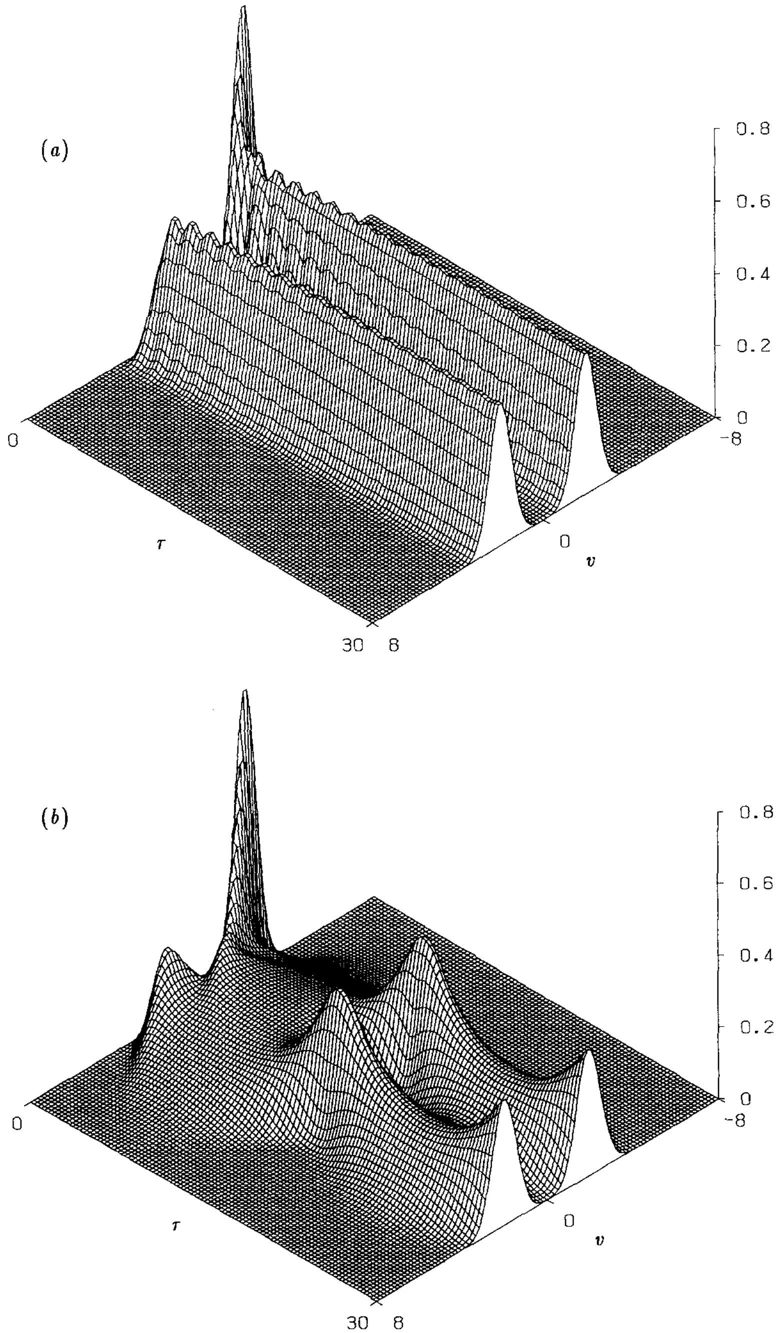


Fig. 7. (a) Approach to equilibrium for a Lorentz gas ($m_A \to 0$) under the influence of a periodic external field $a(\tau) = a \cos \omega\tau$ for $a = 1$ and $\omega = 2$. The equilibrium distribution becomes field independent and is determined by the even part of the initial distribution. (b) Same as (a), with lower frequency $\omega = 0.2$. Three regimes can be distinguished: (1) collisional changes determine the shape of the distribution function, (2) diffusive motion dominates (as in a constant field; compare Fig. 4), (3) slowly decaying oscillating approach to a field-independent equilibrium distribution.

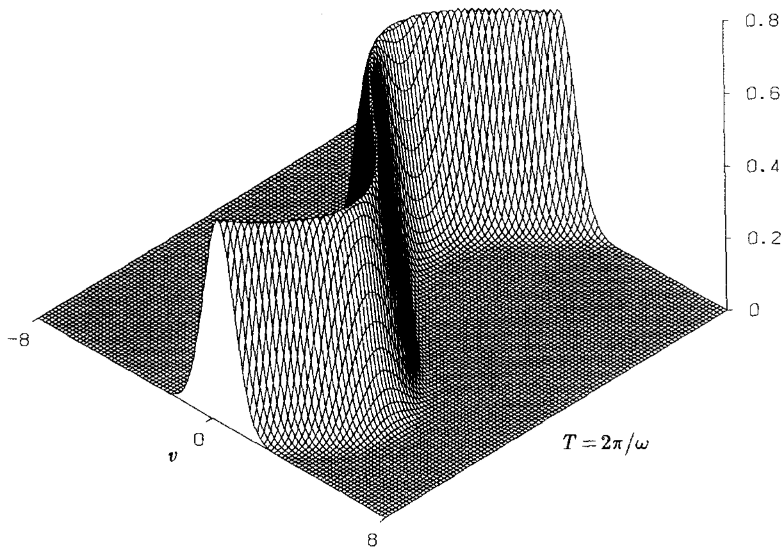


Fig. 8. Stationary velocity distribution function for a Rayleigh gas ($m_A \to \infty$) under the influence of a periodic external field $a(\tau) = a \cos \omega\tau$ for $a = 24$ and $\omega = 8$. The distribution remains a Gaussian with constant variance and periodic mean.

## 7. CONCLUSION

The aim of this paper was to calculate the influence of a constant and a time-dependent external field on a mixture of Maxwellian particles with different masses in one dimension. The results are summarized in Table I. For the constant field case the well-knon difference between a hard rod and a Maxwellian system should be stressed. In the latter case there is no qualitative difference in the mean and the variance if one compares high- and low-field limits. The most striking result for the constant field case is shown in Fig. 3, where the strong deviation from the Gaussian shape can be seen.

The limiting case of the Lorentz gas ($m_A \to 0$) behaves in a surprising way for both the constant and the periodic field. In the former case the velocity distribution becomes diffusive and spreads out indefinitely; in the latter case it becomes independent of the external field. The opposite limit, the Rayleigh gas ($m_A \to \infty$), behaves as expected in the simplest fashion. The stationary distribution is a periodically shifted Gaussian with constant variance. For the general case of finite mass ratios there is a rich spectrum of distribution functions and the details depend strongly on the ratio of the field strength and the frequency of the field.

**Table I. Overview of the Long Time Behavior of the Distribution Functions for Different External Fields and Different Mass Ratios**[a]

| | $m_A \to 0$ (Lorentz gas) | $m_A$ finite | $m_A \to \infty$ (Rayleigh gas) |
|---|---|---|---|
| No field $a(\tau) \equiv 0$ | $h^{eq}(v) = h_{ev,0}(v)$ | $h^{eq}(v) = f_A(v)$ | $h^{eq}(v) = f_A(v)$ |
| Constant field $a(\tau) \equiv a(>0)$ | $\lim_{\tau \to \infty} h(v, \tau) = 0$ (diffusive process) | $m_A = m_B$: $h^{eq}(v) = \frac{1}{a} f_A(v) \exp\left[\left(\frac{1}{2a} - v\right)^2\right] \operatorname{erfc}\left(\frac{1}{2a} - v\right)$ $m_A \neq m_B$: $h^{eq}(v)$ given by Eq. (4.7) or (4.12) | $h^{eq}(v) = f_A(v - a)$ |
| Periodic field $a(\tau) = a \cos \omega\tau$ | $h^{eq}(v) = h_{ev,0}(v)$ | $m_A = m_B$: Periodic $h^s(v, \tau)$ given by Eq. (6.3) $m_A \neq m_B$: Periodic $h^s(v, \tau)$ given by Eq. (6.9) | $h^s(v, \tau) = f_A(v - v_e(\tau))$, periodic $v_e(\tau)$ given by Eq. (6.13) |

[a] $h_{ev,0}$ is the even part of the initial distribution, $h^{eq}$ is the equilibrium distribution, $h^s$ is the stationary distribution, and $f_A$ is the Maxwell–Boltzmann equilibrium distribution.

## APPENDIX

In this Appendix we derive a representation of the powers $x^n$ in terms of the Hermite polynomials $H_n(x)$. This representation is useful for calculating the moments of a function given as a series of Hermite polynomials. Since $H_n(x)$ is even (odd) when $n$ is even (odd), we can treat the cases $n=2m$ and $n=2m+1$ separately.

For $n=2m$ we find[21]

$$H_{2m}(x)=\sum_{k=0}^{m} a_{m,k}x^{2k} \tag{A1a}$$

with

$$a_{m,k}=\frac{(-1)^{m+k}(2m)!\,2^{2k}}{(m-k)!\,(2k)!}, \qquad k\leqslant m \tag{A1b}$$

We are looking for coefficients $b_{m,k}$ so that

$$x^{2m}=\sum_{k=0}^{m} b_{m,k}H_{2k}(x) \tag{A2}$$

where $b_{m,k}=0$ for $k>m$. In the language of linear algebra, we are looking for the inverse matrix $B=(b_{m,k})$ of the lower triangular matrix $A=(a_{m,k})$. Therefore we have to solve

$$\sum_{l=k}^{m} a_{m,l}b_{l,k}=\delta_{m,k} \tag{A3}$$

From Eq. (A3) the coefficients $b_{m,k}$ can be calculated successively. For $k=m$ one finds immediately $b_{m,m}=1/a_{m,m}=2^{-2m}$. Putting $k=m-1$, $m-2$, $m-3$,..., successively one is lead to the conjecture that

$$b_{m,k}=\frac{(2m)!}{2^{2m}(m-k)!\,(2k)!}, \qquad k\leqslant m \tag{A4}$$

In order to prove Eq. (A4), it remains to be shown that

$$\sum_{l=k}^{m}\frac{(-1)^l}{(m-l)!\,(l-k)!}=0 \qquad \text{for} \quad k<m \tag{A5}$$

and this can be shown by binomially expanding the function $f(x)=(1-x)^m$, differentiating both sides $k$ $(<m)$ times, and putting $x=1$.

By the same arguments one can show that the inversion of

$$H_{2m+1}(x) = (-1)^m (2m+1)! \sum_{k=0}^{m} \frac{(-1)^k 2^{2k+1}}{(m-k)!\,(2k+1)!} x^{2k+1} \qquad \text{(A6)}$$

is given by

$$x^{2m+1} = \frac{(2m+1)!}{2^{2m+1}} \sum_{k=0}^{m} \frac{1}{(m-k)!\,(2k+1)!} H_{2k+1}(x) \qquad \text{(A7)}$$

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